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A new class of perfect Hoàng graphs

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Abstract

A Hoàng graph $G = (V, E)$ is a graph whose edges may be coloured with two colours (red and white) such that no induced path on four vertices has wings of the same colour. Hoàng (1995) conjectured that these graphs are perfect and proved two partial results concerning their perfectness. We use here a different technique to establish that for *reducible Hoàng graphs* this conjecture is true.

1. Introduction

Berge [1] defined a graph G to be *perfect* if, for every induced subgraph H of G, the chromatic number of H is equal to the size of a largest clique in H. He also formulated the following still open conjecture, called the strong perfect graph conjecture (abbreviated SPGC):

Conjecture 1 (SPGC). A graph is perfect if and only if it contains no odd hole and no odd antihole.

A *hole* is a chordless cycle of length at least four, while an *antihole* is the complement graph of a hole. A hole (or an antihole) is *odd* if it has an odd number of vertices, and *even* if it has an even number of vertices. Both the odd holes and odd antiholes are *minimal imperfect graphs*, that is, they are not perfect, although all their proper subgraphs are. Moreover, these are the only minimal imperfect graphs known at the present time.

Since all attempts to prove the SPGC have failed, the generally accepted compromise is to reduce the difficulty of the problem by restricting it to special cases, i.e. to particular classes of graphs. The class we are interested in this paper consists of

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reducible Hoàng graphs. The meanings of the terms not defined here may be found in [2].

Let P_4 denote a path on four vertices (note that P_4 has three edges and, therefore, is an odd path). If P_4 has vertex set $\{a, b, c, d\}$ and edge set $\{ab, bc, cd\}$, then ab and cd are called its *wings*. A graph $G = (V, E)$ is said to be a *Hoàng graph* if its edges may be coloured with two colours R (red) and W (white) in such a way that the wings of any P_4 are of different colours. The problem of proving that the Hoàng graphs are perfect has been formulated by Hoàng [6], who also gave two partial results:

Theorem (Hoàng [6]). *If the edges of a graph may be coloured with* R *and* W *such that*

(i) *the wings of any induced P_4 are differently coloured,*

(ii) *every edge which is the middle of an induced P_4 is coloured* W,

then the graph is perfect.

Theorem (Hoàng [6]). *If a graph admits an edge colouring in* R *and* W *such that in every induced P_4 and in every induced C_4 the non-adjacent edges have different colours, then it is a perfect graph.* (C_4 *denotes a cycle of length four*).

The results we present here extend the list of these subclasses of perfect graphs. The supplementary condition we impose in this case does not concern the colouring (as for the subclasses considered by Hoàng), but the forbidden induced subgraphs.

2. Reducible Hoàng graphs

As we shall see, the class of reducible Hoàng graphs is strongly related to the class of gem-free graphs, where we call *gem* the graph in Fig. 1. It is quite easy to notice that there is also a kind of resemblance between Hoàng graphs and Raspail graphs (see the definition below). In fact, the reducible Hoàng graphs are included in neither of these classes, nor include any of them, but it is worth noticing these affinities since they allow us to solve different parts of our problem using one similarity or the other one.

The class of graphs we define in this section — and whose perfectness will be subsequently proved — has a very useful property: any graph of this class is either 'simple' (i.e. both Hoàng graph and gem-free graph), or decomposable into several subgraphs in the same class. If it is 'simple', then proving its perfection becomes an easy matter, because of the similarities indicated above. Otherwise, the decomposition operation ensures that the graph is perfect if and only if the corresponding subgraphs are perfect.

We shall say that a Hoàng graph $G = (V, E)$ is *reducible* if it does not contain as an induced subgraph any of the graphs F_1, F_2, F_3, F_4 in Fig. 2.

The remaining part of this section is devoted to the perfectness of reducible Hoàng graphs. The proof has two steps.

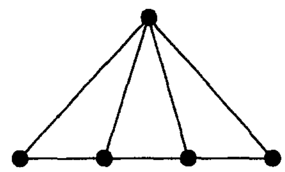

Fig. 1.

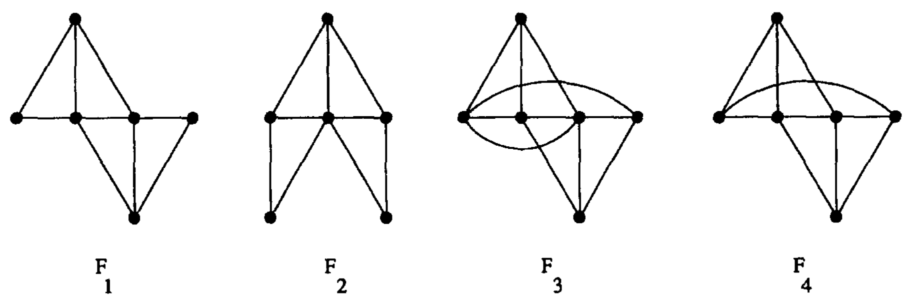

Fig. 2.

Firstly, we make use of the well-known result of Chvátal [3] (independently found by Olaru [8]) asserting that minimal imperfect graphs are *unbreakable*, that is, neither the graph nor its complement contains a disconnecting set with a universal vertex (called a *star-cutset*). As a consequence, any *breakable* graph (i.e. which is not unbreakable) is either perfect or contains a proper subgraph which is minimal imperfect. Our result ensures that:

Theorem 1. *A reducible Hoàng graph is either breakable or gem-free.*

Secondly, we prove the following theorem, which is sufficient to deduce the perfectness of reducible Hoàng graphs:

Theorem 2. *If G is a gem-free Hoàng graph, then G is perfect.*

The immediate corollary of the two results above is:

Corollary. *Every reducible Hoàng graph is perfect.*

Indeed, if there existed an imperfect reducible Hoàng graph G, then it should contain a minimal imperfect subgraph G' which should also be a reducible Hoàng graph. By Theorem 1, this minimal imperfect graph should be either breakable (and this is not possible, by Chvatal's result) or gem-free (and, in this case, by Theorem 2, we have that G' is perfect).

Before starting the reasoning, let us introduce some specific notation. The *neighbourhood* $N(v)$ of the vertex v is the set of vertices adjacent to v; its *non-neighbourhood* $M(v)$ is defined as $M(v) = V(G)\backslash N(v)$. Given an arbitrary vertex v and a colour $\mathrm{B} \in \{\mathrm{R}, \mathrm{W}\}$, a vertex $q \in N(v)$ is called a *vertex coloured* B (with respect to v) and is denoted by $q(\mathrm{B})$ if vq is B. A set $Q \subseteq N(v)$ is called *monocoloured* if all the vertices in Q have the same colour with regard to v. It is called *bicoloured* if it contains vertices of different colours with regard to v.

Now, we are able to present the proofs of the theorems. If $X \subseteq V(G)$, denote by $[X]_G$ the subgraph of G induced by the vertices in X.

Proof of Theorem 1. Obviously, for any reducible Hoàng graph G, exactly one of the following statements holds:

(i) either for every $v \in V$, all the components of $[N(v)]_G$ are monocoloured,

(ii) or there exist at least one vertex w whose neighbourhood graph $[N(w)]_G$ contains a bicoloured component.

As we shall prove, the first possibility implies almost immediately that G is gem-free (Lemma 1), while the second one ensures that G is breakable (Lemma 2). □

Lemma 1. *If G is a Hoàng graph such that, for every $v \in V$, $[N(v)]_G$ has only monocoloured components, then G is gem-free.*

Proof. Suppose there is a vertex $v \in V$ such that the neighbourhood of v contains the induced P_4 denoted by $abcd$. Then a, b, c, d are in the same component of $[N(v)]_G$, consequently all the edges va, vb, vc, vd have the same colour, say R. Among ab and cd, one of the edges is coloured W, say ab. Then the neighbourhood of the vertex a contains a bicoloured component, namely the one which contains b and v, a contradiction. □

The reasoning we use to prove the second part of the theorem is based on an idea of Hayward [5] which is concretized as follows:

Claim. *Let G be a reducible Hoàng graph and v be a vertex of G. If*
Q is a bicoloured connected subgraph of $[N(v)]_G$,
P is a connected subgraph of $[M(v)]_G$,
then
(C1) *either there is a vertex q in Q adjacent to no vertex in P,*
(C2) *or there is a vertex p in P adjacent to all the vertices in Q.*

The proof of this claim is long (although elementary), so we omit it. It is worth noticing that the forbidden subgraphs in Fig. 2 are used essentially all along the reasoning.

Once this result is proved, it is quite easy to deduce the next one:

Lemma 2. *If in G there exists a vertex v such that* $[N(v)]_G$ *has at least one bicoloured component, then G is breakable.*

Proof. If $[N(v)]_G$ is connected, then by the preceding claim we have (C1) or (C2) with $Q = [N(v)]_G$, $P = [M(v)]_G$, since the non-neighbourhood graph $[M(v)]_G$ of v may be supposed connected (otherwise we already have a star-cutset).

In the case that (C1) is true, $\{v\} \cup N(v) \setminus \{q\}$ is a star-cutset of G. If (C2) is true, then $\{p\} \cup N(v)$ is a star-cutset of G, since $M(v) \neq \{p\}$ (otherwise $\bar{G}$ would be non-connected and, therefore, breakable).

We may then suppose that there exist at least two components in $[N(v)]_G$. Let N_1 be the bicoloured component and N_2 another component. In N_1 we can find vertices y(R) and z(W) which are adjacent. Consider also a vertex x of N_2.

If $\{y\} \cup N(y) \setminus \{z\}$ is not a star-cutset, there exists a chordless path joining x and z which does not contain vertices in that set. Let $x' \in N_2$ be the first vertex of the path (when going from x to z) such that its neighbour w on the path is in $[M(v)]_G$. Then $yvx'w$ implies that $x'w$ is W, while $zvx'w$ implies $zw \in E$. Then from $x'wzy$ we deduce that zy is R.

In a similar way, if we consider the set $\{z\} \cup N(z) \setminus \{y\}$ which is not a star-cutset, we obtain that zy should be coloured W. We have a contradiction. □

From Lemmas 1 and 2 we can easily deduce Theorem 1. □

Proof of Theorem 2. To state this result, we shall use the similarity of the Hoàng graphs with Raspail graphs. Recall that a graph $G = (V, E)$ is called a *Raspail graph* if every odd cycle of G contains at least one short chord, i.e. a chord joining two vertices at distance 2 along the cycle. A simple generalization yields the notion of a quasi-Raspail graph: a graph G is *quasi-Raspail* if for every vertex v and every odd chordless path P of length at least three in $G - v$ joining two vertices $x, y \in N(v)$, the cycle with vertices $\{v\} \cup V(P)$ has at least one short chord. Obviously, a Raspail graph is a quasi-Raspail graph, but the converse is not true, as shown by $\bar{C}_7$.

Lemma 3. *Any Hoàng graph is quasi-Raspail.*

Proof. Suppose the contrary and let $G = (V, E)$ be a Hoàng graph that contradicts the lemma, that is, it contains a vertex v and an odd chordless path P (joining the vertices $x, y \in N(v)$) such that the odd cycle with vertices $\{v\} \cup V(P)$ has no short chords. Since the cycle is odd and a Hoàng graph contains no induced odd cycle, there exists a pair (a, b) of vertices on P, both adjacent to v, such that $ab \in E(P)$, $a \in P_{xb}$ and no other similar pair is contained in P_{xa}. Analogously, there is a pair (a', b') of vertices on P, both adjacent to v, such that $a'b' \in E(P)$, $b' \in P_{a'y}$ and no other similar pair is contained in $P_{b'y}$ (the two pairs are possibly identical).

Let also c, d be the neighbours of x, respectively of a on P_{xa}, and c', d' the neighbours of y, respectively of b' on $P_{yb'}$. Since the cycle has no short chords, $a \neq c$ and $b' \neq c'$.

Two cases could occur:

Case 1: vx, vy have the same colour (say R). From $yvxc$ and $yvad$ we obtain that cx, ad are W, while from $xvyc'$ and $xvb'd'$ we have that yc', $b'd'$ are also coloured W. If $c \neq d$ and e is the other neighbour of c on the path, then $vxce$ implies that ce is W (if ce was R, then we should have $ev \in E$ and $cevy$ would yield a contradiction). So, independently of whether $c = d$ or not, the first two edges of the path are coloured W, and the same holds for the last two edges. Together with the fact that G is a Hoàng graph, that means the path is even, a contradiction.

Case 2: vx is R, vy is W. Once more, $yvxc$ and $yvad$ imply that xc and ad are R, while $xvyc'$ and $xvb'd'$ give that $b'd'$ and yc' are W. The graph induced by d, a, v, x implies $c = d$. The first two edges of the path are therefore coloured R. In the same way we deduce that the last two edges are W, consequently, again, the path is even, a contradiction. □

Concerning the perfectness of Raspail graphs, Lubiw proved the following result (a graph G is P_4-free if it does not contain a P_4 as an induced subgraph):

Theorem (Lubiw [7]). *A minimal imperfect Raspail graph does not contain a vertex v whose neighbourhood graph $[N(v)]_G$ is P_4-free.*

That we can extend to quasi-Raspail graphs as below:

Lemma 4. *A minimal imperfect quasi-Raspail graph does not contain a vertex v whose neighbourhood $[N(v)]_G$ is P_4-free.*

The proof of this lemma follows the same steps as the proof of Lubiw's theorem, with slight modifications specific to quasi-Raspail graphs.

Now, to prove Theorem 2 from these two lemmas is an easy matter. An arbitrary gem-free Hoàng graph G is either perfect or contains a minimal imperfect graph G'. In the latter case, we first apply Lemma 3 to deduce that G' is quasi-Raspail, and then Lemma 4 to obtain a contradiction. □

3. Final remarks

Despite their apparent simplicity, the Hoàng graphs seem to require at least a little improvement of the methods we possess to prove perfectness. The idea is supported both by Hoàng's approach (based on a result of Chvátal and Sbihi [4] on *homogeneous pairs*, a technical generalization of the well-known homogeneous sets) and by our own approach (which also needs a generalization of a well-known result, as well as some technical steps).

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